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TRANSONIC LIMITS OF LINEARIZED THEORY

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I. INTRODUCTION*

The transonic regime, extending from Mach numbers at which shock waves first appear with the associated drag rise to Mach numbers at which the head shock wave is firmly attached to the nose of the body, is well known. Not so well known, perhaps, is the reason for the failure of linearized theory to describe the flow in this regime, in particular, to permit an accurate calculation of the pressure.

Linearized theory is identical in its physical content and mathematical formulation with the classical theory of acoustics.** It is based on the conservation laws plus the assumption that all disturbances are so small that squares and higher powers of disturbance quantities can be neglected. The fundamental result of the theory is that disturbances propagate as waves with a constant speed $c = \sqrt{\gamma \frac{p_0}{\rho_0}}$ relative to the air at rest. In aerodynamic applications the small disturbances are caused by thin bodies or wings moving through the air, and, further, the disturbances vary as the thickness of the body. Thus, for sufficiently thin bodies the theory might seem always valid. However, a body in flight through air at rest is continually emitting disturbance waves which radiate to the sides and, in general, escape to the front or rear of the body. In the special case when the body is flying at the sonic speed c, some of the waves remain with the body and cause the pressure to build up with time. For example, according to the linearized theory, the pres-

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**For example, see Ref. 1, Vol. II, p. 107, where the basic formula for steady or unsteady motion of non-lifting wings is presented.

sure on a slender body of revolution flying steadily at sonic speed grows as $\log t$; t = time. Thus no steady state exists for steady flight at Mach number $M = 1$, according to linearized theory. It follows that linearized theory is a poor approximation for steady flight in a range of Mach numbers about 1, the transonic regime.

Transonic theory is an improvement in that it accounts for the fact that the speed of disturbances is not constant but varies depending on local conditions. Mathematically this is done by retaining certain important terms proportional to the square of small disturbances. Thus the instantaneous speed of disturbances near a body changes as the instantaneous pressure distribution on the body, and a steady state solution at $M = 1$ is possible. Shock waves, which can be disregarded in linearized theory, must be considered in transonic theory as there is a possibility of small disturbances overtaking one another to produce a steep front. Unfortunately, the introduction of the non-linearities of transonic theory does away with the possibility of representing the flow by addition of waves (or solutions) and increases the difficulty in obtaining solutions enormously. (For an example of a solution see Ref. 2.)

A last remark concerns bodies accelerating through sonic speed. Although the emitted waves no longer remain with the body, transonic effects may still be important. Only for a sufficiently large acceleration can linearized theory be used to compute the pressures.

In the following sections some of the points mentioned here will be shown in more detail by exhibiting the equations of motion which include linearized and transonic theories. For various cases, the computations of linearized theory will be used to estimate the non-linear effect, or the size of the squared terms. When these terms become appreciable a computation based on linearized theory is no longer sufficiently accurate and transonic theory must be used.

II. SECOND-ORDER EQUATIONS OF MOTION. LINEARIZED AND TRANSONIC THEORIES

Consider a system of coordinates (x, y, z, t) in which the air is at rest at infinity with a pressure $P = P_0$, density $\rho = \rho_0$. Then, considering small disturbances but including the squared terms, it can be shown that a velocity potential $\phi(x, y, z, t)$ exists (for details see Ref. 3), which completely describes the flow and satisfies a generalized wave equation:

$$\frac{1}{c^2}\left\{1 + (\gamma - 1)\frac{\phi_t}{c^2}\right\}\phi_{tt} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\phi)^2 - \Delta\phi = 0 \tag{1}$$

with

$$\vec{q} = \text{fluid velocity vector} = \nabla\phi = (\phi_x, \phi_y, \phi_z) \tag{2}$$

$$s = \text{condensation} = \frac{\rho - \rho_0}{\rho_0} = -\frac{1}{c^2}\phi_t - \frac{1}{2c^2}\left\{(\nabla\phi)^2 + (\gamma - 2)\frac{\phi_t^2}{c^2}\right\} \tag{3}$$

$$p = \text{pressure perturbation} = \frac{P - P_0}{P_0} = \gamma s + \frac{\gamma(\gamma - 1)}{2}s^2 \tag{4}$$

$$c = \text{sonic speed at infinity} = \sqrt{\frac{\gamma P_0}{\rho_0}} \tag{5}$$

If all the squared terms are dropped out of Eqs. 1 - 4, then the classical equations of acoustics are obtained with the constant wave speed c.

If Eq. 1 is considered in the more usual coordinates, namely those fixed in a body moving at a constant speed U in the negative x-direction, and if the flow is steady in the new coordinate system, we have

$$\left\{\begin{array}{l} \bar{x} = x + Ut \\ \bar{t} = t \\ \bar{y} = y \\ \bar{z} = z \end{array}\right\} \quad \frac{\partial}{\partial t} \to U\frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial y} \to \frac{\partial}{\partial \bar{y}} \quad \text{etc.} \tag{6}$$

and

$$\left\{1 - M^2 - \frac{(\gamma-1)M^2 + 2}{c^2} U\phi_{\bar{x}}\right\}\phi_{\bar{x}\bar{x}} + \phi_{\bar{y}\bar{y}} + \phi_{\bar{z}\bar{z}} = \frac{U}{c^2}(\phi_{\bar{y}}^2 + \phi_{\bar{z}}^2)_{\bar{x}} \tag{7}$$

$$= 0 \tag{7'}$$

Dropping out squared terms in Eq. 7 results in the usual linearized equation of steady subsonic or supersonic motion in the $\bar{x}$-direction, which is thus equivalent to the classical acoustic equation. Dropping out only the right-hand side as in Eq. 7', and retaining the important non-linear terms containing $\phi_{\bar{x}}\,\phi_{\bar{x}\bar{x}}$ results in the usual transonic equation (cf. Ref. 4 and Ref. 5); for $M \doteq 1$, $\frac{(\gamma-1)M^2+2}{c^2} \doteq \frac{\gamma+1}{a^*}M^2$. This non-linear term includes the effect of local Mach number in the equation for we have the approximate relation:

$$1 - M^2_{LOCAL} \doteq 1 - M^2 - \frac{\gamma+1}{a^*}M^2\phi_{\bar{x}} \tag{8}$$

If $\phi_{\bar{y}\bar{y}} + \phi_{\bar{z}\bar{z}}$ is thought of as the net outflow of fluid across the y, z faces of an infinitesimal cube $dx\,dy\,dz$, the non-linear term describes the flow as accelerating if it is locally supersonic ($M_{LOCAL} > 1$, $\phi_{\bar{x}\bar{x}} > 0$) and decelerating if it is subsonic ($M_{LOCAL} < 1$, $\phi_{\bar{x}\bar{x}} < 0$). Thus the famous sonic throat is included in the transonic equation.

It is interesting now to see what role the non-linearity plays in the general equation (1), or more specifically, in its transonic version, with the x -axis as the flight direction:

$$\frac{1}{c^2}\left\{1 + (\gamma-1)\frac{\phi_t}{c^2}\right\}\phi_{tt} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi_x^2 - \Delta\phi = 0 \tag{9}$$

The effect of the non-linearity in modifying the wave speed can be seen from the characteristics or wave surfaces for infinitesimal disturbances that go with Eq. 9. If only (x, t) are considered, the characteristics are given by

$$\frac{dx}{dt} \doteq \left(\phi_x \mp \frac{\gamma-1}{2c}\phi_t\right) \pm c \tag{10}$$

with similar results for more space dimensions. The dependence of wave speed on local conditions, and the possibility of forming steep fronts is shown by Eq. 10. The same phenomenon appears as throats at the sonic velocity when viewed in the moving coordinate system $(\bar{x}, t)$.

III. EXAMPLES FOR STEADY FLIGHT

In this section the non-linear effect will be examined for several simple cases in order to show the dependence on different parameters which enter. Linearized theory is a computation based on the assumption that all quadratic terms are negligible compared to linear ones. When the solution of linearized theory shows that the neglected term is appreciable, the limit of its validity has certainly been reached and a more complete theory must be used.

A measure of the non-linear effect is given by the ratio of the main non-linear terms to a significant linear one. The breakdown of linearized theory at $M = 1$ is due to the vanishing of $(1 - M^2)\phi_{\bar{x}\bar{x}}$. Hence, in the transonic range, consider

$$\lambda = \frac{\text{non-linear term}}{\text{linear term}} \doteq \frac{(\gamma + 1) M^2 \phi_{\bar{x}}}{1 - M^2} \tag{11}$$

If $\lambda = 1$ near a body the non-linear terms are appreciable in a region, while for $\lambda < 1$, say, linearized theory is a good approximation. For example, for steady two-dimensional flow past an airfoil of thickness ratio δ, linearized theory gives the pressure coefficient:

$$c_p = -2\frac{\phi_{\bar{x}}}{U} \sim \frac{\delta}{\sqrt{1 - M^2}} \tag{12}$$

Using Eq. 12 in Eq. 11 gives

$$\lambda \sim \frac{(\gamma + 1) M^2 \delta}{(1 - M^2)^{3/2}} = \xi_\infty^3 \tag{13}$$

ξ_∞ = transonic similarity parameter

which shows the relation of λ to the conventional similarity parameter. Thus, linearized theory is no longer a good approximation when $\lambda \sim \xi_\infty^3 \sim 1$.

It might be remarked that the $\lambda = 1$ corresponds, according to Eq. 8, to an estimate of the critical Mach number of subsonic flow, when the flow is locally sonic.

As the first example, consider a symmetric parabolic arc body of revolution of thickness ratio δ flying at subsonic Mach number M. A simple application of slender body version of linearized theory (e. g., Eq. 21, Ref. 6) gives the result that at the maximum thickness

$$\lambda = (\gamma + 1) M^2 \frac{\delta^2}{1 - M^2}\left\{2 \log \frac{2}{\delta\sqrt{1 - M^2}} - 3\right\} \tag{14}$$

The simple proportionality of Eq. 12 no longer holds, but the thickness ratio enters as a square times a log. The result is typical for smooth bodies of revolution and gives an estimate of when to expect transonic effects. Fig. 1 gives a plot of Eq. 14 for several thickness ratios. It is shown, for example, that a .10 thick body begins to have transonic troubles at $M \doteq .95$. Slender body theory can be extended to bodies of arbitrary cross-section. From the general formulas, for example of Ref. 6, it is easily seen that λ can be reduced if the cross-section area is given as uniform a distribution as possible along the body.

Next, consider a wing of rectangular planform and symmetric parabolic arc airfoil section of thickness ratio δ. An elementary calculation of the necessary source distribution of linearized theory (cf. Ref. 7) gives the result:

$$\lambda = \frac{4}{\pi}(\gamma+1)\frac{M^2}{(1-M^2)^{3/2}}\,\delta\left[AR\sqrt{1-M^2}\, sh^{-1}\frac{1}{AR\sqrt{1-M^2}}\right] \qquad (15)$$

with $AR =$ aspect ratio. As would be expected from the general similarity laws (Ref. 5), the significant parameters are M, thickness ratio δ, and effective aspect ratio $AR\sqrt{1-M^2}$. In contrast to the body of revolution case, the thickness ratio enters as a first power, and the AR in a complicated way. For $AR\sqrt{1-M^2} \gg 1$, Eq. 15 reduces to the two-dimensional case, since $sh^{-1}\frac{1}{AR\sqrt{1-M^2}} \doteq \frac{1}{AR\sqrt{1-M^2}}$ (cf. Eq. 13). Thus, one can write Eq. 15 as

$$\frac{\lambda}{\lambda_2} = AR\sqrt{1-M^2}\, sh^{-1}\frac{1}{AR\sqrt{1-M^2}} \qquad (15')$$

$\lambda_2 =$ two dimensional value of λ

For $AR\sqrt{1-M^2} \ll 1$, $sh^{-1}\frac{1}{AR\sqrt{1-M^2}} \doteq \log\frac{2}{AR\sqrt{1-M^2}}$ and a similarity with Eq. 14 is obvious considering AR behaves like δ. Fig. 2 is a plot of Eq. 15 for $\delta = .10$ with λ evaluated at the center of the wing. Since $\lambda \sim \delta$, it is seen, as is well known, that reducing the thickness is more effective than reducing the aspect ratio for delaying transonic effects.

A similar calculation has been made for a delta wing with the same airfoil section, and the results for λ at the center of the wing are plotted in Fig. 3.

For supersonic flow there is a corresponding limitation on linearized theory as M decreases toward 1. For two-dimensional flow the limit is again near $\xi_\infty = 1$ with $\xi_\infty = \frac{[(\gamma+1)\delta]^{1/3}}{\sqrt{M^2-1}}$. A three-dimensional transonic effect can be illustrated for the case of rectangular wing with a simple wedge airfoil. Transonic theory allows for the fact that the Mach number is reduced and the corresponding Mach angle increased behind a shock wave. As a result the region of influence of the tips is enlarged compared with the linearized calculation. The effect is shown in Fig. 4, which shows the minimum aspect ratio for wing-tip independence in similarity form.

IV. EXAMPLE OF ACCELERATION THROUGH SONIC SPEED

By considering the translation of axes (Eq. 6), a corresponding λ can be defined for unsteady motion:

$$\lambda = \frac{\frac{\gamma-1}{c^4}\phi_t \phi_{tt} + \frac{2}{c^2}\phi_x \phi_{xt}}{\frac{1}{c^2}\phi_{tt} - \phi_{xx}} \tag{16}$$

which reduces to the previous λ in the case of steady flow. Again, it is a measure of the non-linear effect. To evaluate λ , the solution to the linearized equation

$$\frac{1}{c^2}\phi_{tt} - \phi_{xx} = \phi_{rr} + \frac{1}{r}\phi_r \tag{17}$$

must be computed. For a body of revolution in arbitrary motion along a straight path, the potential can be represented by a distribution of sources along that path (Fig. 5), and the general slender body formula can be developed (Ref. 3). We have, if $S(x,t)$ is the source strength,

$$\phi(x,r,t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{S\left(\xi, t - \frac{\sqrt{(x-\xi)^2 + r^2}}{c}\right)d\xi}{\sqrt{(x-\xi)^2 + r^2}} \tag{18}$$

$$\cong \frac{1}{2\pi} S(x,t) \operatorname{LOG} r - \frac{1}{4\pi}\left\{S(x_1, t - \frac{x - x_1}{c}) \operatorname{LOG}(x - x_1)\right.$$

$$\left. + S(x_2, t - \frac{x_2 - x}{c}) \operatorname{LOG}(x_2 - x)\right\}$$

$$- \frac{1}{4\pi}\int_{x_1}^{x_2}\left\{\frac{1}{c} S_t(\xi, t - \frac{|x-\xi|}{c}) + \operatorname{SGN}(x-\xi)\, S_x(\xi, t - \frac{|x-\xi|}{c})\right\} \operatorname{LOG}|x-\xi|\, d\xi$$

$$- \frac{1}{4\pi}\int_{x_3}^{x_4} \frac{S(\xi, t - \frac{|x-\xi|}{c})}{|x-\xi|}\, d\xi + O(r^2 \operatorname{LOG} r) \tag{19}$$

The significance of $(x_1, x_2)(x_3, x_4)$ is indicated in Fig. 7. Eq. 19 is the general slender body formula, an asymptotic expansion valid close to the body for small r . It includes the steady slender body formulas for subsonic and supersonic motion (Eq. 10 and Eq. 21 of Ref. 6) as special cases obtained by the translation of axes. For an arbitrary cross-section area distribution along the axis $A(x,t)$, the source strength according to Eq. 19 is

$$S(x,t) = \frac{\partial A}{\partial t} \tag{20}$$

For the special case of a body of given shape moving along the axis, $A(x,t) = A(x + \int_0^t U(\tau)d\tau)$, and Eq. 20 becomes

$$S(x,t) = U(t)A'(x + \int_0^t U(\tau)d\tau) \qquad (21)$$

$U(t)$ = speed of body in negative x-direction

$A(x)$ = cross-section area of body

The fundamental result (Eq. 21) is due to Frankl (Ref. 8).

As an example, consider a symmetric parabolic arc body of thickness ratio δ, which has a uniform acceleration $2b$ and passes through sonic speed. Applying Eqs. 19 and 21 gives the result:

$$\lambda = \frac{3}{4}(\gamma + 1)\frac{\delta^2}{\sqrt{\frac{b\ell}{c^2}}}\left\{\log \frac{2}{\delta^2}\sqrt{\frac{c^2}{b\ell}} - \frac{9}{4}\right\} \qquad (22)$$

where

ℓ = length of body

$\frac{b\ell}{c^2}$ = dimensionless acceleration parameter

when evaluated at the maximum thickness at the instant of passing through sonic speed. The form of Eq. 22 is identical with λ for the steady case (Eq. 14) if $\sqrt{\frac{b\ell}{c^2}} \leftrightarrow 1 - M^2$. From Eq. 22 it can be seen that for a fixed δ it is always theoretically possible to accelerate through sonic speed so fast that $\lambda < 1$ and linearized theory is valid. However, a plot of Eq. 22, shown in Fig. 7, indicates that this is not often practically possible.

As a side result, the calculated pressure distribution at sonic speed is shown in Fig. 8 for $\delta = .10$, $\frac{b\ell}{c^2} = .026$. The slight unsymmetry of the pressure distribution results in a drag:

$$C_D \doteq 2.9\,\delta^2 + O(\sqrt{\delta}) \tag{23}$$

V. CONCLUDING REMARKS

The previous sections were intended to show the relationship of transonic and linearized theories, and to give quantitative estimates of the non-linear effects in several simple cases. The results indicate the importance of transonic theory for practical applications even in the case of unsteady motion.

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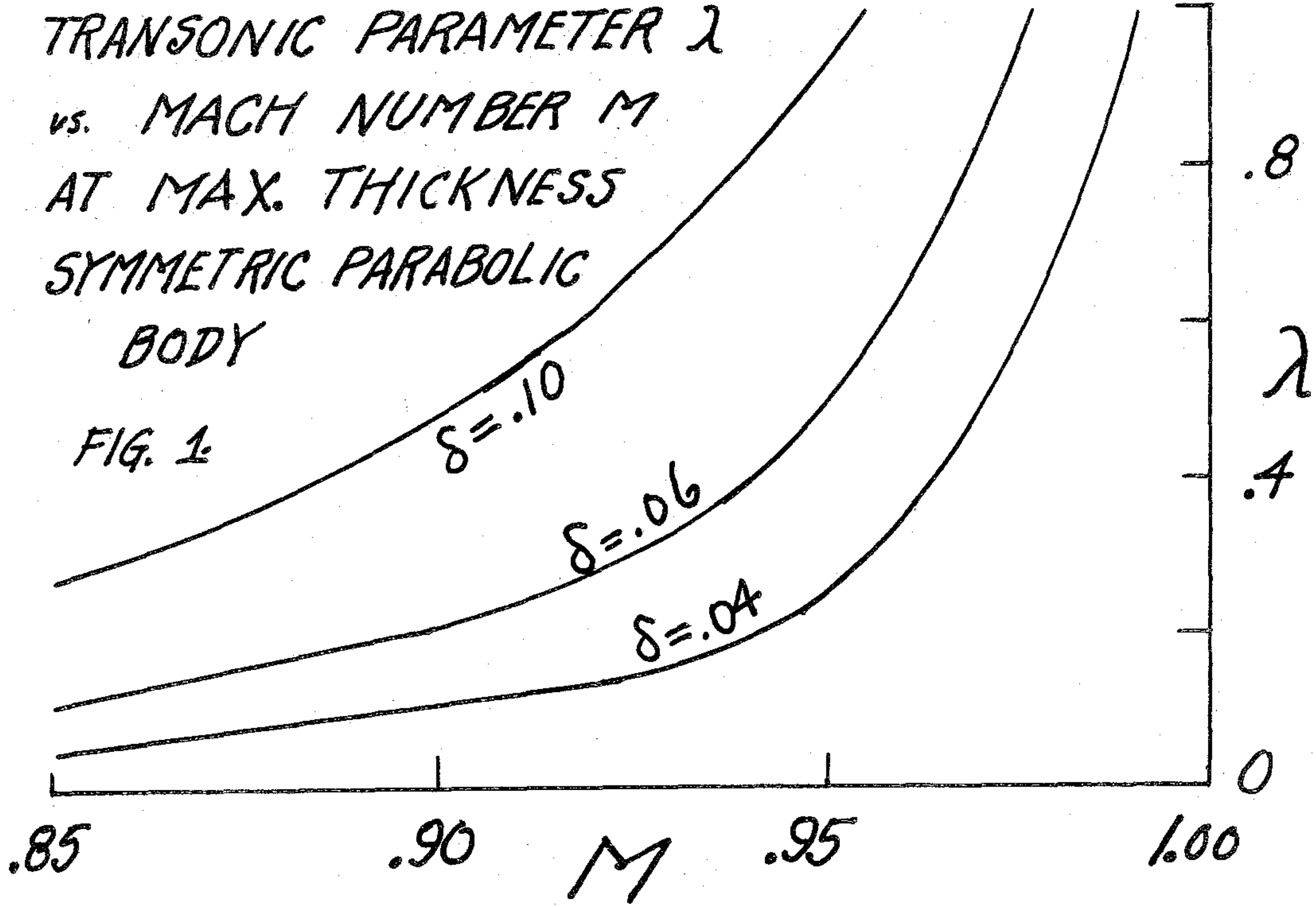


TRANSONIC PARAMETER λ vs. MACH NUMBER M AT MAX. THICKNESS SYMMETRIC PARABOLIC BODY

FIG. 1.

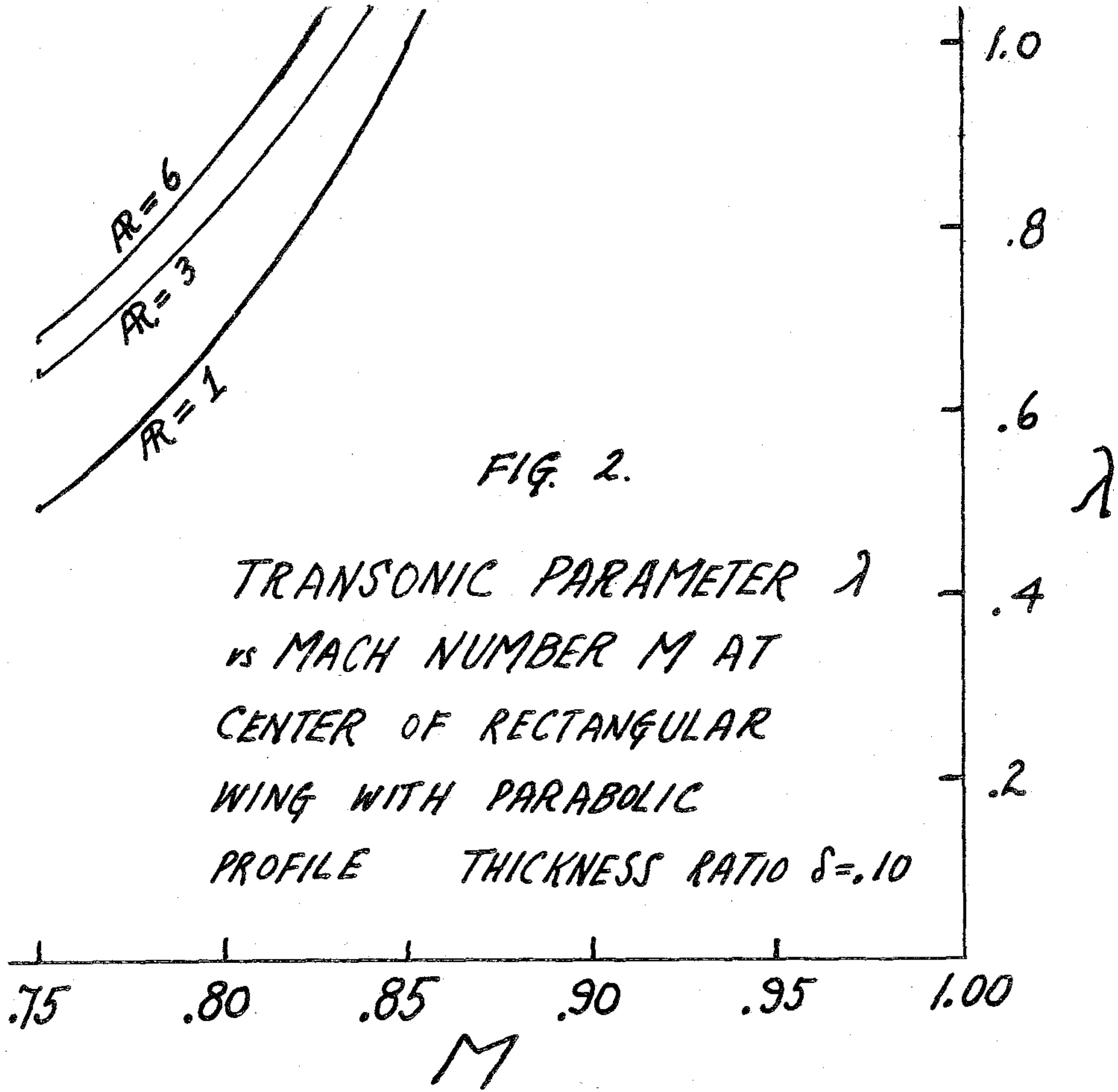


FIG. 2.

TRANSONIC PARAMETER λ vs MACH NUMBER M AT CENTER OF RECTANGULAR WING WITH PARABOLIC PROFILE THICKNESS RATIO $\delta = .10$

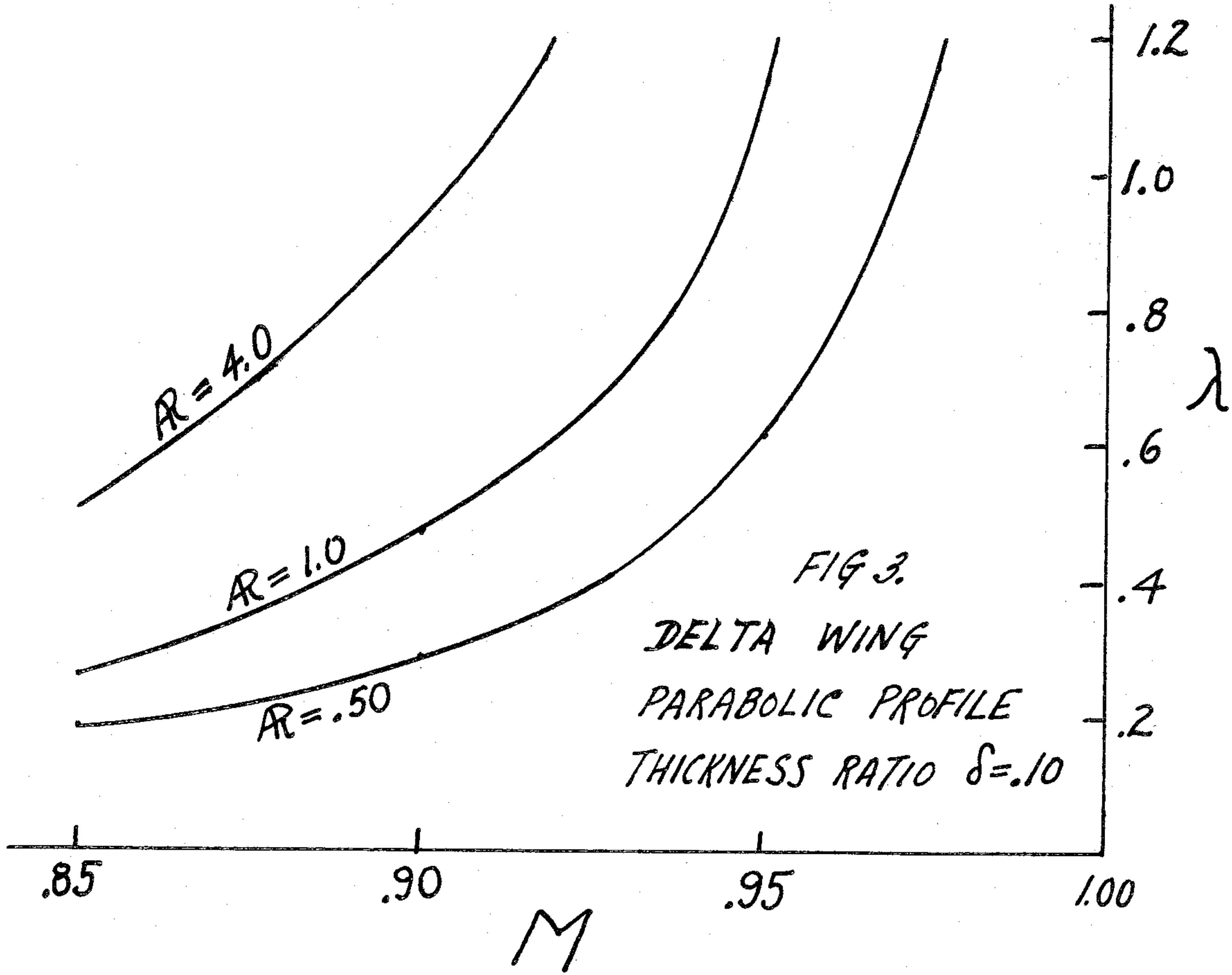
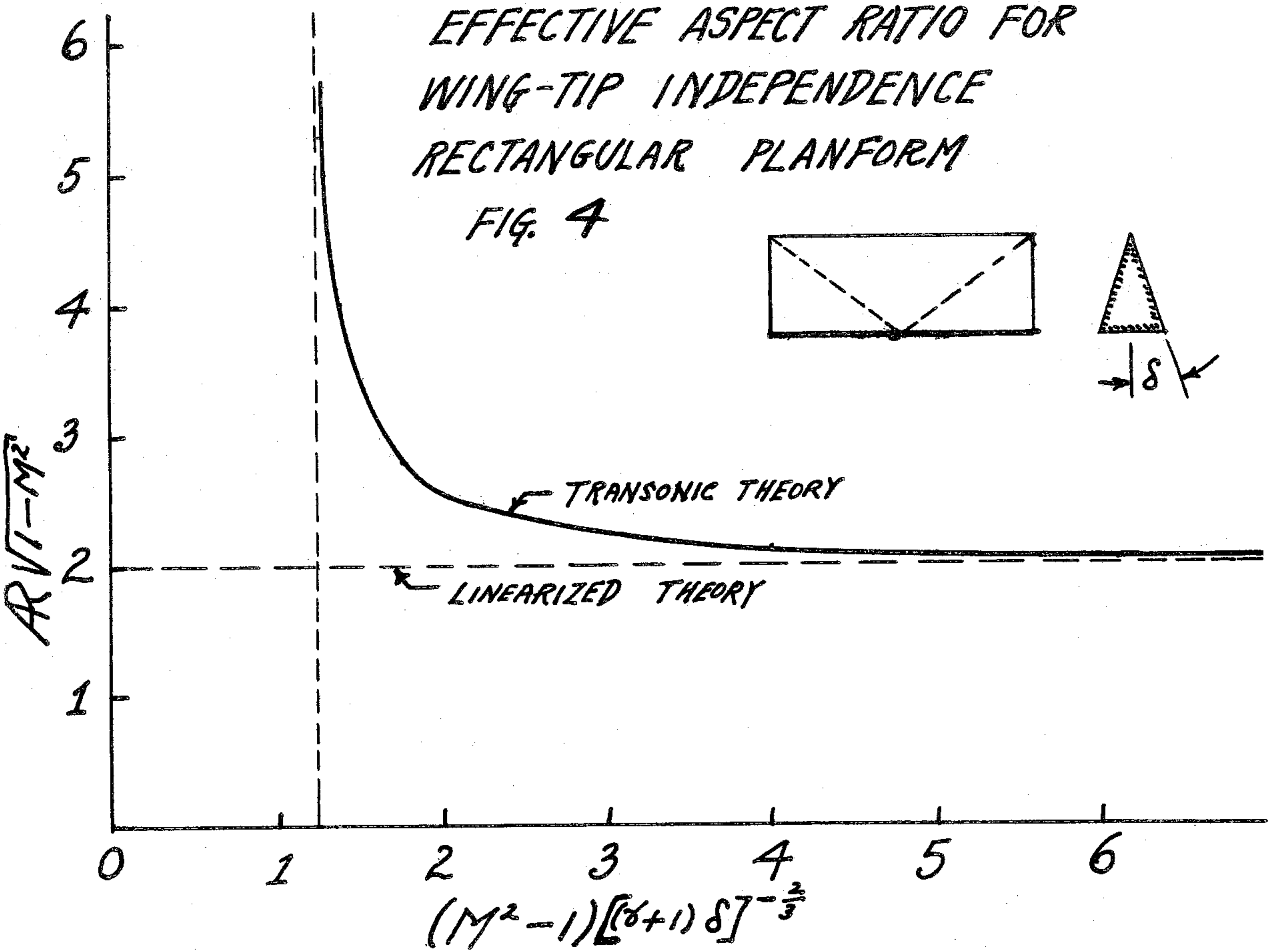


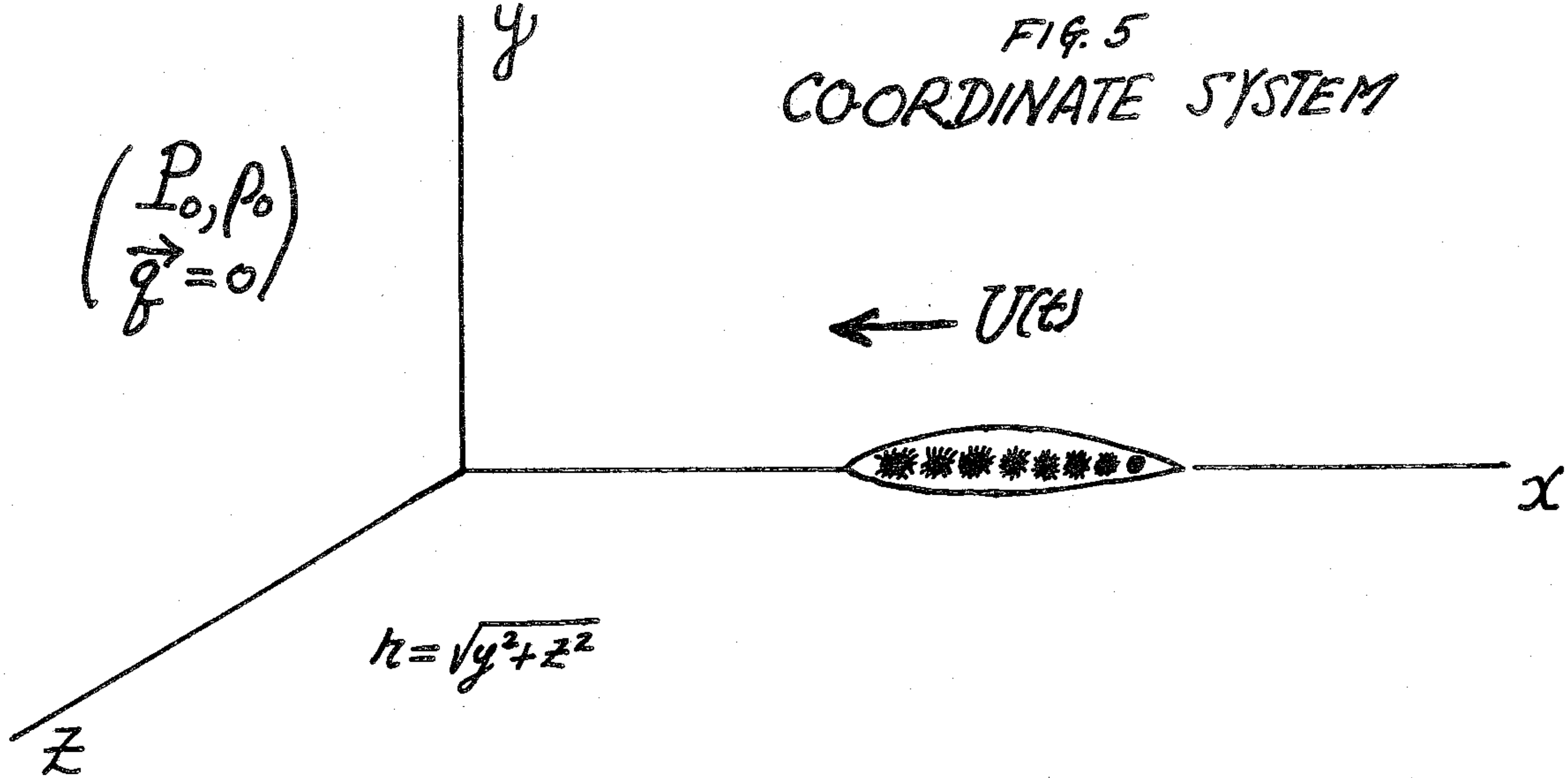
FIG 3.
DELTA WING
PARABOLIC PROFILE
THICKNESS RATIO $\delta = .10$

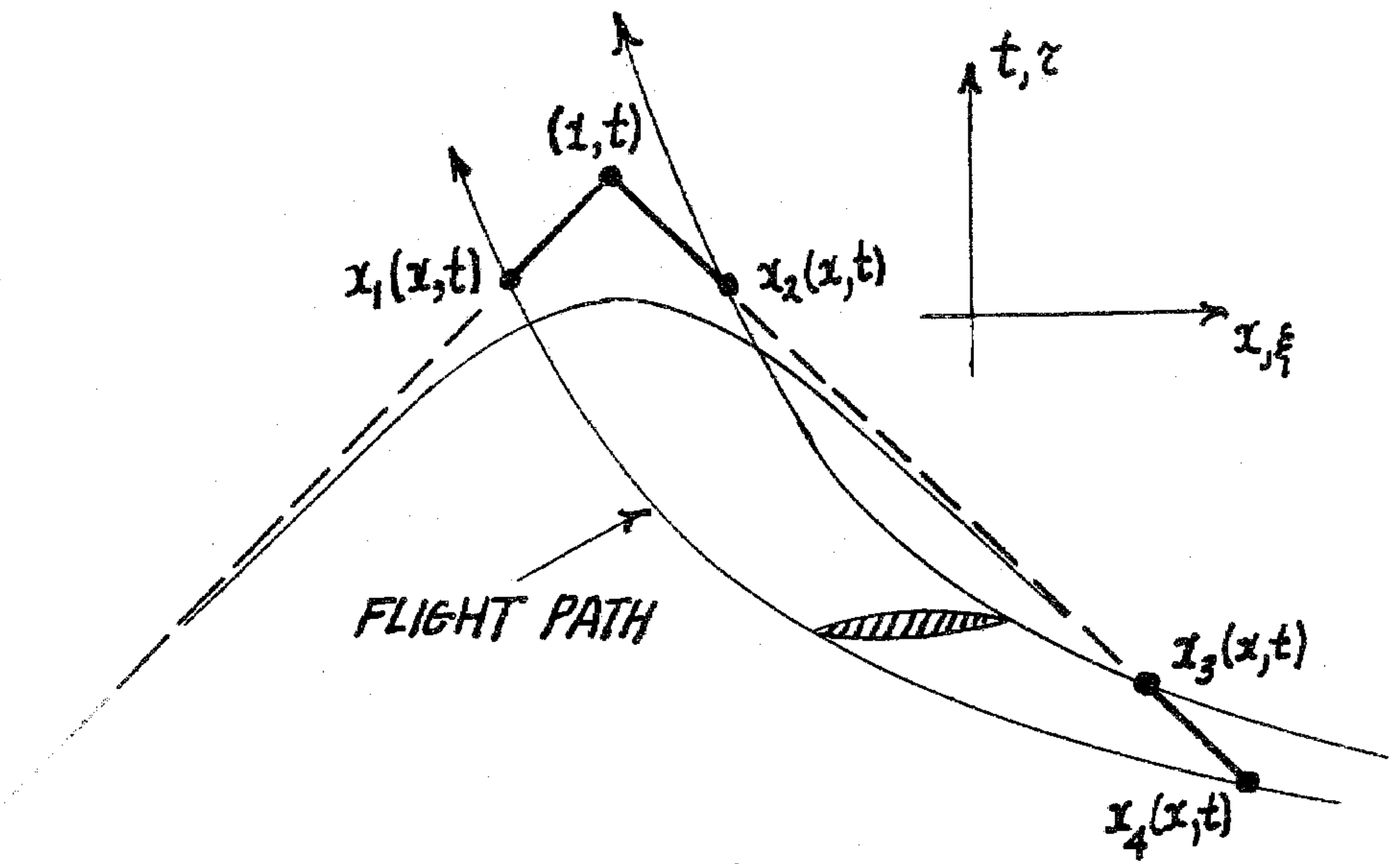


EFFECTIVE ASPECT RATIO FOR WING-TIP INDEPENDENCE RECTANGULAR PLANFORM

FIG. 4

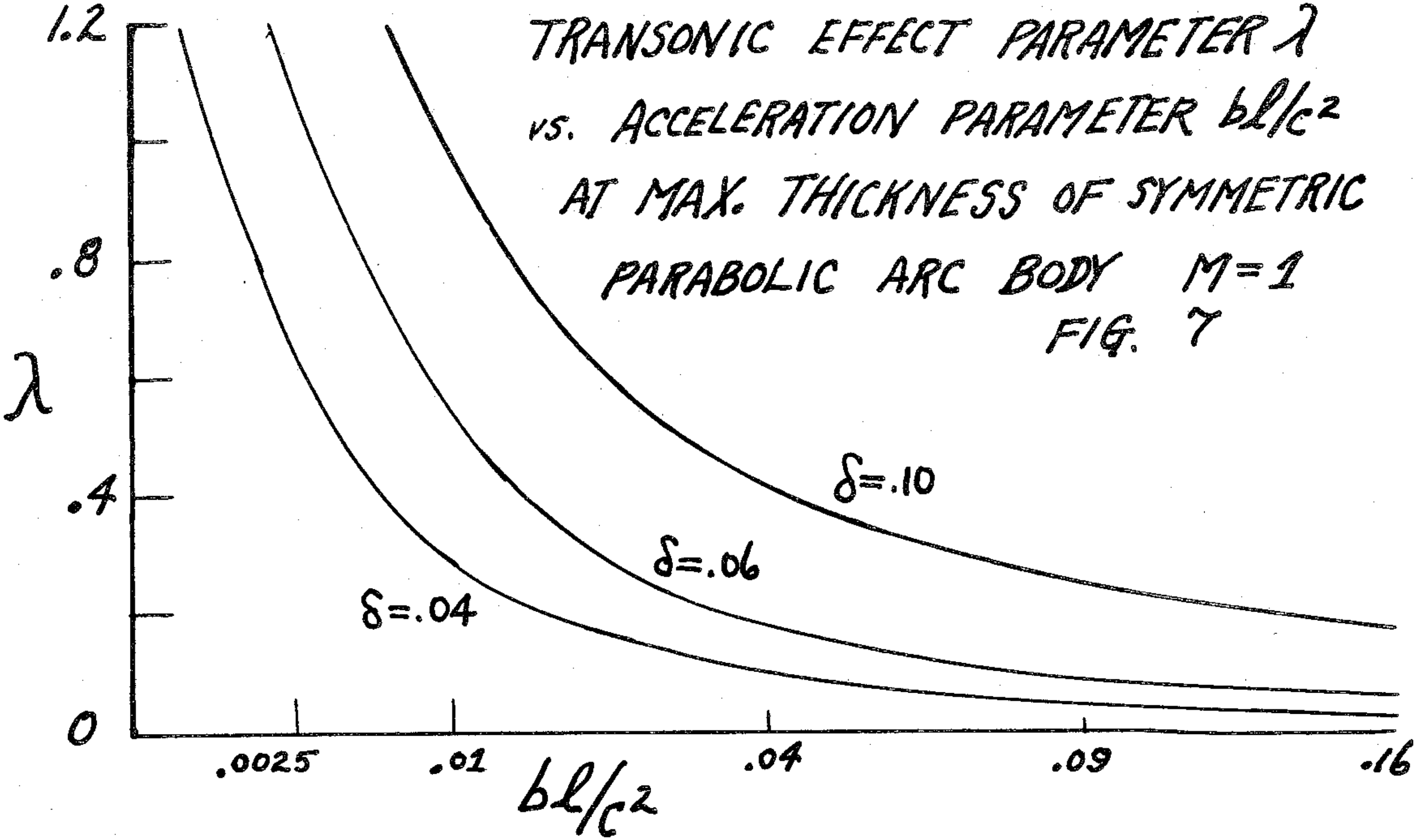
FIG. 5
COORDINATE SYSTEM





SPACE-TIME DIAGRAM

FIG. 6



TRANSONIC EFFECT PARAMETER λ vs. ACCELERATION PARAMETER $b\ell/c^2$ AT MAX. THICKNESS OF SYMMETRIC PARABOLIC ARC BODY $M = 1$

FIG. 7